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An exact formula for general spectral correlation function of random Hermitian matrices

Yan V Fyodorov[1] **and Eugene Strahov**

Department of Mathematical Sciences, Brunel University, Uxbridge, UB8 3PH, UK

E-mail: Yan.Fyodorov@brunel.ac.uk and Eugene.Strahov@brunel.ac.uk

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Abstract
We have found an exact formula expressing a general correlation function containing both products and ratios of characteristic polynomials of random Hermitian matrices. The answer is given in the form of a determinant. An essential difference from the previously studied correlation functions (of products only) is the appearance of non-polynomial functions along with the orthogonal polynomials. These non-polynomial functions are the Cauchy transforms of the orthogonal polynomials. The result is valid for arbitrary ensemble of $\beta = 2$ symmetry class and generalizes recent asymptotic formulae obtained for Gaussian unitary ensemble and its chiral counterpart by different methods.

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1. Introduction

A classical result of analysis known from the nineteenth century is that orthogonal polynomials can be represented as multi-variable integrals. For example, let $\pi_j(x)$ denote a monic orthogonal polynomial of jth degree with respect to the measure $d\mu(x) = \exp(-NV(x))\,dx$ on the real axis,

$$\int \pi_j(x)\pi_k(x)\,e^{-NV(x)}\,dx = c_j c_k \delta_{jk} \tag{1}$$

and $\pi_j(x) = x^j +$ lower degrees. Then for the monic orthogonal polynomial $\pi_N(x)$ there exists an integral representation (see [1])

$$\pi_N(x) = \frac{1}{Z_N}\int \prod_{j=1}^{N}(x - x_j)\,e^{-N\sum_{j=1}^{N}V(x_j)}\,\triangle^2(x_1,\ldots,x_N)\,dx_1\ldots x_N. \tag{2}$$

[1] On leave from Petersburg Nuclear Physics Institute, Gatchina 188350, Russia.

In the above formula $\triangle(x_1, \ldots, x_N)$ stands for the Vandermonde determinant, and the constant Z_N is given by the product $Z_N = N! \prod_{j=0}^{N-1} c_j^2$.

The classical integral representation of the monic orthogonal polynomial $\pi_N(x)$ given in (2) suggests a random matrix interpretation. Indeed if we consider an ensemble of random $N \times N$ Hermitian matrices H with the joint probability density of eigenvalues (see [2])

$$\mathcal{P}^{(N)}(x_1, \ldots, x_N) = \frac{1}{Z_N} \exp\left\{-N \sum_{i=1}^{N} V(x_i)\right\} \triangle^2(x_1, \ldots, x_N) \tag{3}$$

the monic orthogonal polynomial $\pi_N(x)$ can be understood as an average of the *characteristic polynomial* $\mathcal{Z}_N[x, H] = \det(x - H)$ over the ensemble

$$\pi_N(x) = \langle \mathcal{Z}_N[x, H] \rangle_H. \tag{4}$$

A natural question which arises at this point is the following: what do we obtain if instead of one characteristic polynomial we average a combination of characteristic polynomials (for example, their product). This question was recently addressed by Brezin and Hikami [3] and Mehta and Normand [4] who derived a generalization of formula (4). The authors considered the ensemble average of products of characteristic polynomials. It was found that this average is expressed as a determinant:

$$\left\langle \prod_{j=1}^{L} \mathcal{Z}_N[\lambda_j, H] \right\rangle_H = \frac{1}{\Delta(\hat{\lambda})} \det \begin{vmatrix} \pi_N(\lambda_1) & \pi_{N+1}(\lambda_1) & \ldots & \pi_{N+L-1}(\lambda_1) \\ \pi_N(\lambda_2) & \pi_{N+1}(\lambda_2) & \ldots & \pi_{N+L-1}(\lambda_2) \\ \vdots & & & \\ \pi_N(\lambda_L) & \pi_{N+1}(\lambda_L) & \ldots & \pi_{N+L-1}(\lambda_L) \end{vmatrix} \tag{5}$$

where $\Delta(\hat{\lambda}) \equiv \Delta(\lambda_1, \ldots, \lambda_L)$ (see also a related paper by Forrester and Witte [5] which deals with a particular case of positive integer moments: $\lambda_1 = \cdots = \lambda_L$).

In the present paper we obtain a further generalization of formulae (4) and (5). Namely, we compute the general spectral correlation function of characteristic polynomials defined as

$$\mathcal{K}_N(\hat{\epsilon}, \hat{\mu}) = \left\langle \frac{\prod_{j=1}^{L} \mathcal{Z}_N[\mu_j, H]}{\prod_{j=1}^{M} \mathcal{Z}_N[\epsilon_j, H]} \right\rangle_H. \tag{6}$$

Here the symbols $\hat{\epsilon}$, $\hat{\mu}$ denote the vectors with the components:

$$\hat{\epsilon} = (\epsilon_1, \ldots, \epsilon_M) \qquad \hat{\mu} = (\mu_1, \ldots, \mu_L). \tag{7}$$

For the function $\mathcal{K}_N(\hat{\epsilon}, \hat{\mu})$ to be well defined we assume that $\operatorname{Im} \epsilon_k \neq 0$, and we also assume that $N \geqslant M$, $N \geqslant L$. Then the result we obtain for the correlation function $\mathcal{K}_N(\hat{\epsilon}, \hat{\mu})$ is

$$\mathcal{K}_N(\hat{\epsilon}, \hat{\mu}) = \frac{\prod_{j=N-M}^{N-1} \gamma_j}{\Delta(\hat{\mu})\Delta(\hat{\epsilon})} \det \begin{vmatrix} h_{N-M}(\epsilon_1) & h_{N-M+1}(\epsilon_1) & \ldots & h_{N+L-1}(\epsilon_1) \\ \vdots & & & \\ h_{N-M}(\epsilon_M) & h_{N-M+1}(\epsilon_M) & \ldots & h_{N+L-1}(\epsilon_M) \\ \pi_{N-M}(\mu_1) & \pi_{N-M+1}(\mu_1) & \ldots & \pi_{N+L-1}(\mu_1) \\ \vdots & & & \\ \pi_{N-M}(\mu_L) & \pi_{N-M+1}(\mu_L) & \ldots & \pi_{N+L-1}(\mu_L) \end{vmatrix} \tag{8}$$

where $h_k(\epsilon)$ denotes the Cauchy transform of the monic orthogonal polynomial $\pi_k(x)$,

$$h_k(\epsilon) = \frac{1}{2\pi \mathrm{i}} \int \frac{\pi_k(x)\, \mathrm{e}^{-NV(x)}\, \mathrm{d}x}{x - \epsilon} \qquad \operatorname{Im} \epsilon \neq 0 \tag{9}$$

and

$$\gamma_{n-1} = -\frac{2\pi \mathrm{i}}{c_{n-1}^2}. \tag{10}$$

The above formula is of interest since it shows the place of the correlation function $\mathcal{K}_N(\hat{\epsilon}, \hat{\mu})$ in the theory of orthogonal polynomials. Furthermore, it provides new insight on $\mathcal{K}_N(\hat{\epsilon}, \hat{\mu})$, for example, it relates $\mathcal{K}_N(\hat{\epsilon}, \hat{\mu})$ with Riemann–Hilbert problem for orthogonal polynomials (see section 5).

Correlation functions $\mathcal{K}_N(\hat{\epsilon}, \hat{\mu})$ are interesting objects themselves since they contain a very detailed information about spectra of random matrices. In particular, knowledge of such correlations enables us to extract the n-point correlation function of spectral densities. Moreover, distributions of some interesting physical quantities e.g., level curvatures are expressed in terms of $\mathcal{K}_N(\hat{\epsilon}; \hat{\mu})$ (see, e.g., [6] and appendix A of [7] for more examples).

The investigation of correlation functions of characteristic polynomials and their moments is also motivated by a hope to relate statistics of zeros of the Riemann zeta function to that of eigenvalues of large random matrices [8, 3]. The spectral determinants are also relevant for interesting combinatorial problems (see [9]). Other numerous applications of spectral determinants are in the theory of quantum chaotic and disordered systems, and in quantum chromodynamics (see [10, 11] for an extensive list of references).

There are several analytical techniques for dealing with the integer moments (positive or negative) of characteristic polynomials and more general correlations functions. Their applicability varies with the nature of the underlying random matrix ensemble. The particular case of the Gaussian measure—Gaussian unitary ensemble (GUE), as well as its chiral counterpart, chGUE—can be studied very efficiently by a modification of the standard supersymmetry technique [12] suggested recently by the authors [10, 11, 13]. One starts, as usual, by representing each of the characteristic polynomials as a Gaussian integral over anti-commuting (Grassmann) variables and inverse characteristic polynomial by Gaussian integral over complex variables. This allows us to average the resulting expressions straightforwardly. At the next step one employs the so-called Hubbard–Stratonovich transformation for 'fermionic' degrees of freedom, and then exploits the Itzykson–Zuber–Harish–Chandra integrals together with their natural non-compact extensions. A detailed account of the method and results as well as related references can be found in the recent papers [10–13].

The resulting expressions in [11, 13] revealed a very attractive determinantal structure which was especially evident in the case of chiral GUE (Laguerre ensemble) [13] (see also [14]). When one deals with the positive moments only, such structures naturally arise in the framework of the orthogonal polynomial method [3, 4]. This fact is suggestive of the idea that the determinantal structure in the general case (i.e. when both products and ratios of characteristic polynomials are involved) should be valid for an arbitrary unitary-invariant potential. In this paper we show that this is indeed the case.

The paper is organized as follows. In section 2 we derive an algebraic identity which represents $\prod_{j=1}^{M} \mathcal{Z}_N^{-1}[\epsilon_j, H]$ as a sum over permutations. This identity together with formula (5) will permit us to rewrite $\mathcal{K}_N(\hat{\epsilon}, \hat{\mu})$ as an integral over M variables with subsequent derivation of formula (8). Section 3 illustrates our approach on the simplest case $M = 1$ and $L = 0$. Here we derive a multi-variable integral representation for Cauchy transforms of monic orthogonal polynomials. The computation of the general correlation function $\mathcal{K}_N(\hat{\epsilon}, \hat{\mu})$ for any integers $0 \leqslant L, M \leqslant N$ is given in section 4. In section 5 we establish relation with the Riemann–Hilbert problem for orthogonal polynomials proposed by Fokas, Its and Kitaev [15, 16] (see also [17, 18]), and outline a way to investigate the large N asymptotic of $\mathcal{K}_N(\hat{\epsilon}, \hat{\mu})$ by Deift–Zhou steepest descent/stationary phase method [18–23]. A detailed asymptotic analysis of

$\mathcal{K}_N(\hat{\epsilon}, \hat{\mu})$ is beyond the scope of this paper and will be given in the subsequent publication [24]. Finally in section 6 we briefly discuss some 'duality relations' for matrix integrals emerging for the case of the Gaussian potential.

2. Inverse of products of characteristic polynomials as a sum over permutations

Let $M \leqslant N$ and $x_1, \ldots, x_N$ denote the eigenvalues of a matrix H ($\dim H = N \times N$). Then the following algebraic identity holds:

$$\prod_{l=1}^{M} \frac{\epsilon_l^{N-M}}{\mathcal{Z}_N[\epsilon_l, H]} = \sum_{\sigma \in \mathsf{S}_N/\mathsf{S}_{N-M} \times \mathsf{S}_M} \left(\prod_{i,j=1}^{M} \frac{x_{\sigma(i)}^{N-M}}{\epsilon_{\sigma(j)} - x_{\sigma(i)}} \right) \times \frac{\triangle(x_{\sigma(1)}, \ldots, x_{\sigma(M)}) \triangle(x_{\sigma(M+1)}, \ldots, x_{\sigma(N)})}{\triangle(x_{\sigma(1)}, \ldots, x_{\sigma(N)})} \tag{11}$$

where S_N is the permutation group of the indices $1, \ldots, N$, S_M is the permutation group of the first M indices and S_{N-M} is the permutation group of the remaining $N - M$ indices. Identity (11) follows as a consequence of the Cauchy–Littlewood formula [25]

$$\prod_{j=1}^{M} \prod_{i=1}^{N} (1 - x_i y_j)^{-1} = \sum_{\lambda} s_\lambda(x_1, \ldots, x_N) s_\lambda(y_1, \ldots, y_M) \tag{12}$$

and the Jacobi–Trudi identity [25]:

$$s_\lambda(x_1, \ldots, x_N) = \frac{\det\left(x_i^{\lambda_j - j + N}\right)}{\triangle(x_1, \ldots, x_N)} \tag{13}$$

where the Schur polynomial $s_\lambda(x_1, \ldots, x_N)$ corresponds to the partition λ and the indices i, j take the values from 1 to N. In order to prove (11) we rewrite the Schur polynomial $s_\lambda(x_1, \ldots, x_N)$ in (12) as

$$s_\lambda(x_1, \ldots, x_N) = \frac{\sum_{\pi \in \mathsf{S}_N} (-)^{\nu_\pi} x_{\pi(1)}^{\lambda_1 - 1 + N} \ldots x_{\pi(M)}^{\lambda_M - M + N} x_{\pi(M+1)}^{-M-1+N} \ldots x_{\pi(N)}^{0}}{\triangle(x_1, \ldots, x_N)}. \tag{14}$$

(As $s_\lambda(y_1, \ldots, y_M) = 0$ for any partition with the number of rows larger than M only the partitions with $\lambda_{M+1} = \cdots = \lambda_N = \cdots = 0$ contribute in equation (12).) In equation (14) $\nu_\pi = 1$ (or $\nu_\pi = 0$) if the permutation $\pi \in \mathsf{S}_N$ is odd (even).

Any permutation $\pi \in \mathsf{S}_N$ in the above sum can be decomposed as a product of three subsequent permutations, i.e. $\pi = \sigma \cdot \pi_2 \cdot \pi_1$. The first one, $\pi_1 \in \mathsf{S}_M$, is a permutation of elements of the set $[1, 2, \ldots, M]$; the second permutation, $\pi_2 \in \mathsf{S}_{N-M}$, is that of elements of the set $[M + 1, M + 2, \ldots, N]$; and the third permutation, $\sigma \in \mathsf{S}_N/\mathsf{S}_M \times \mathsf{S}_{N-M}$, is an exchange of elements between these two sets. For example, let us take $M = 3$, $N = 5$ and permutation $\pi = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 2 & 5 & 3 \end{smallmatrix}\right)$. This permutation can be represented as follows:

$$\pi = \sigma \cdot \pi_2 \cdot \pi_1$$

$$\pi_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 2 & 4 & 5 \end{pmatrix} \qquad \pi_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 5 & 4 \end{pmatrix} \qquad \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 3 & 4 & 2 \end{pmatrix}.$$

We now rewrite the denominator on the right-hand side of equation (14) as

$$\begin{aligned} \triangle(x_1, \ldots, x_N) &= (-)^{\nu_\pi} \triangle(x_{\pi(1)}, \ldots, x_{\pi(N)}) \\ &= (-)^{\nu_\pi} \triangle(x_{\pi(1)}, \ldots, x_{\pi(M)}) \triangle(x_{\pi(M+1)}, \ldots, x_{\pi(N)}) \\ &\quad \times \prod_{i \in [1, \ldots, M], j \in [M+1, \ldots, N]} (x_{\pi(i)} - x_{\pi(j)}) \end{aligned} \tag{15}$$

and observe that

$$\sum_{\pi_1 \in \mathsf{S}_M} \frac{x_{\pi(1)}^{\lambda_1-1+N} \ldots x_{\pi_1(M)}^{\lambda_M-M+N}}{\triangle\big(x_{\pi_1(1)}, \ldots, x_{\pi_1(M)}\big)} = (x_1 \ldots x_M)^{N-M} s_\lambda(x_1, \ldots, x_M) \tag{16}$$

and

$$\sum_{\pi_2 \in \mathsf{S}_{N-M}} \frac{x_{\pi_2(M+1)}^{-M-1+N} \ldots x_{\pi_2(N)}^{0}}{\triangle\big(x_{\pi_2(M+1)}, \ldots, x_{\pi_2(N)}\big)} = 1 \tag{17}$$

After the permutations of types π_1 and π_2 have been performed in equation (14) we remain with the following sum over permutations $\sigma \in \mathsf{S}_N/\mathsf{S}_M \times \mathsf{S}_{N-M}$:

$$\begin{aligned} s_\lambda(x_1, \ldots, x_N) &= \sum_{\sigma \in \mathsf{S}_N/\mathsf{S}_M \times \mathsf{S}_{N-M}} \frac{\big(x_{\sigma(1)}^{N-M} \ldots x_{\sigma(M)}^{N-M}\big) s_\lambda(x_{\sigma(1)} \ldots x_{\sigma(M)})}{\prod_{i\in[1,\ldots,M],\, j\in[M+1,\ldots,N]} (x_{\sigma(i)} - x_{\sigma(j)})} \\ &= \sum_{\sigma \in \mathsf{S}_N/\mathsf{S}_M \times \mathsf{S}_{N-M}} \frac{\big(\prod_{j=1}^M x_{\sigma(j)}^{N-M}\big) s_\lambda(x_{\sigma(1)} \ldots x_{\sigma(M)}) \triangle(x_{\sigma(1)} \ldots x_{\sigma(M)}) \triangle(x_{\sigma(M+1)} \ldots x_{\sigma(N)})}{\triangle(x_{\sigma(1)} \ldots x_{\sigma(N)})}. \end{aligned} \tag{18}$$

We insert expression (18) in the sum in equation (12) and apply the Cauchy–Littlewood formula. It gives

$$\begin{aligned} \frac{1}{\prod_{j=1}^M \prod_{i=1}^N (1 - x_i y_j)} &= \sum_{\sigma \in \mathsf{S}_N/\mathsf{S}_{N-M} \times \mathsf{S}_M} \left(\prod_{i,j=1}^M \frac{x_{\sigma(i)}^{N-M}}{1 - x_{\sigma(i)} y_{\sigma(j)}} \right) \\ &\quad \times \frac{\triangle(x_{\sigma(1)}, \ldots, x_{\sigma(M)}) \triangle(x_{\sigma(M+1)}, \ldots, x_{\sigma(N)})}{\triangle(x_{\sigma(1)}, \ldots, x_{\sigma(N)})}. \end{aligned} \tag{19}$$

It is easy to see that the above formula is equivalent to formula (11).

3. Multi-variable integral representation for Cauchy transforms of orthogonal polynomials

In this section we compute $\big\langle \mathcal{Z}_N^{-1}[\epsilon, H] \big\rangle_H$, $\dim H = N$ and show that this average is equal to $\gamma_{N-1} h_{N-1}(\epsilon)$. Here γ_{N-1} is given by equation (10) and $h_{N-1}(\epsilon)$ is the Cauchy transform of the monic polynomial $\pi_{N-1}(x)$ (equation (9)). The aim is just to illustrate our approach to the computation of the general correlation function $\mathcal{K}_N(\hat{\epsilon}, \hat{\mu})$ of characteristic polynomials on the simple example $L = 0$, $M = 1$.

Once $M = 1$ algebraic identity (11) takes the form

$$\frac{\epsilon^{N-M}}{\mathcal{Z}_N[\epsilon, H]} = \sum_{\sigma \in \mathsf{S}_N/\mathsf{S}_{N-1}} \left(\frac{x_{\sigma(1)}^{N-1}}{\epsilon - x_{\sigma(1)}} \right) \frac{\triangle(x_{\sigma(2)}, \ldots, x_{\sigma(N)})}{\triangle(x_{\sigma(1)}, \ldots, x_{\sigma(N)})}. \tag{20}$$

Averaging the above expression with the probability measure $\mathcal{P}^{(N)}(x_1, \ldots, x_N)$ (equation (3)) we note that each term yields the same contribution ($\mathcal{P}^{(N)}(x_1, \ldots, x_N)$ is symmetric with respect to the permutations). Thus we have

$$\big\langle \mathcal{Z}_N^{-1}[\epsilon, H] \big\rangle_H = \frac{\epsilon^{M-N} N}{Z_N} \int \frac{x_1^{N-1}}{\epsilon - x_1} \triangle(x_2, \ldots, x_N) \triangle(x_1, \ldots, x_N)\, \mathrm{e}^{-N \sum_{i=1}^N V(x_i)}\, \mathrm{d}x_1 \ldots \mathrm{d}x_N. \tag{21}$$

The above integral can be further rewritten as

$$Z_{N-1}\int \mathrm{d}x_1 \frac{x_1^{N-1}\,\mathrm{e}^{-NV(x_1)}}{\epsilon - x_1}\langle \mathcal{Z}_{N-1}[x_1,\tilde{H}]\rangle_{\tilde{H}} \qquad \dim \tilde{H} = N-1. \tag{22}$$

Using formula (4) we then obtain

$$\left\langle \mathcal{Z}_N^{-1}[\epsilon, H]\right\rangle_H = N\epsilon^{N-M}\frac{Z_{N-1}}{Z_N}\int \frac{x^{N-1}\pi_{N-1}(x)\,\mathrm{e}^{-NV(x)}\,\mathrm{d}x}{\epsilon - x}. \tag{23}$$

Now we note that

$$\frac{x^{N-1}}{\epsilon - x} = \frac{x^{N-1}-\epsilon^{N-1}}{\epsilon - x} + \frac{\epsilon^{N-1}}{\epsilon - x}. \tag{24}$$

The first term above is a polynomial of degree $N-2$ which is orthogonal to the polynomial $\pi_{N-1}(x)$. Thus only the second term contributes to the integral. It gives

$$\left\langle \mathcal{Z}_N^{-1}[\epsilon, H]\right\rangle_H = \gamma_{N-1}h_{N-1}(\epsilon) \tag{25}$$

or

$$\gamma_{N-1}h_{N-1}(\epsilon) = \frac{1}{Z_N}\int \prod_{j=1}^{N}(\epsilon - x_j)^{-1}\,\mathrm{e}^{-N\sum_{j=1}^{N}V(x_j)}\triangle^2(x_1,\ldots,x_N)\,\mathrm{d}x_1\ldots x_N \tag{26}$$

where we have used formulae (9) and (10) and have expressed Z_N as $N!\prod_{j=0}^{N-1}c_j^2$.

We expect that the representation of Cauchy transforms of orthogonal polynomials as multi-variable integrals (26) might be known, but we failed to trace it in the standard monographs on the subject.

4. Derivation of the formula for $\mathcal{K}_N(\hat{\epsilon},\hat{\mu})$

Let us now consider the general case. The correlation function $\mathcal{K}_N(\hat{\epsilon},\hat{\mu})$ defined by equation (6) is an integral over N variables $x_1,\ldots,x_N$ (which are eigenvalues of H) with an integrand symmetric under permutations. It then follows that each component of the sum in equation (11) gives the same contribution to the correlation function $\mathcal{K}_N(\hat{\epsilon},\hat{\mu})$. Thus we have

$$\begin{aligned}\mathcal{K}_N(\hat{\epsilon},\hat{\mu}) = \frac{N!}{(N-M)!M!}\prod_{j=1}^{M}\epsilon_j^{M-N}\int \mathrm{d}x_1\ldots \mathrm{d}x_N\mathcal{P}^{(N)}(x_1,\ldots,x_N)\\ \times\left(\prod_{i,j=1}^{M}\frac{x_i^{N-M}}{\epsilon_j - x_i}\right)\frac{\prod_{j=1}^{L}\mathcal{Z}_N[\mu_j,H]}{\prod_{i=1}^{M}\prod_{j=M+1}^{N}(x_i-x_j)}.\end{aligned} \tag{27}$$

We decompose the eigenvalue probability density function as

$$\begin{aligned}\mathcal{P}^{(N)}(x_1,\ldots,x_N) = \frac{Z_{N-M}Z_M}{Z_N}\prod_{i=1}^{M}\prod_{j=M+1}^{N}(x_i-x_j)^2\\ \times\mathcal{P}^{(M)}(x_1,\ldots,x_M)\mathcal{P}^{(N-M)}(x_{M+1},\ldots,x_N).\end{aligned} \tag{28}$$

With the above decomposition the integral expression for the correlation function $\mathcal{K}_N(\hat{\epsilon}, \hat{\mu})$ can be rewritten as

$$\mathcal{K}_N(\hat{\epsilon}, \hat{\mu}) = \frac{N!}{(N-M)!M!} \left(\prod_{j=1}^{M} \epsilon_j^{M-N} \right) \frac{Z_{N-M} Z_M}{Z_N} \\ \times \int \mathrm{d}x_1 \ldots \mathrm{d}x_M \mathcal{P}^{(M)}(x_1, \ldots, x_M) F_{\hat{\epsilon},\hat{\mu}}(x_1, \ldots, x_M) I^{(N-M)}(x_1, \ldots, x_M, \hat{\mu}) \tag{29}$$

where we have introduced two functions of M variables $x_1, \ldots, x_M$. The first function is

$$F_{\hat{\epsilon},\hat{\mu}}(x_1, \ldots, x_M) = \prod_{i,j=1}^{M} \prod_{k=1}^{L} \frac{x_i^{N-M}(\mu_k - x_i)}{\epsilon_j - x_i}. \tag{30}$$

The second function, $I^{(N-M)}(x_1, \ldots, x_M, \hat{\mu})$, can be understood as an averaged value of the products of characteristic polynomials over an ensemble of $(N-M)$-dimensional Hermitian matrices $\tilde{H}$ with eigenvalues $x_{M+1}, \ldots, x_N$,

$$I^{(N-M)}(x_1, \ldots, x_M, \hat{\mu}) = \left\langle \prod_{i=1}^{M} \prod_{k=1}^{L} \mathcal{Z}_{N-M}[x_i, \tilde{H}] \mathcal{Z}_{N-M}[\mu_k, \tilde{H}] \right\rangle_{\tilde{H}}. \tag{31}$$

It is convenient to introduce $(L+M)$-dimensional vector $\hat{r}$ whose components are the integration variables $x_1, \ldots, x_M$ and the elements of the vector $\hat{\mu}$, i.e.

$$\hat{r} = (x_1, \ldots, x_M, \mu_1, \ldots, \mu_L). \tag{32}$$

We exploit formula (5), which for $I^{(N-M)}(x_1, \ldots, x_M, \hat{\mu}) \equiv I^{(N-M)}(\hat{r})$ gives the following expression:

$$I^{(N-M)}(\hat{r}) = \frac{1}{\triangle(\hat{r})} \det[\pi_{N-M+j-1}(r_i)]_{1\leqslant i,j\leqslant L+M}. \tag{33}$$

The Vandermonde determinant $\triangle(\hat{r})$ can be factorized as

$$\triangle(\hat{r}) = \left\{ \prod_{k=1}^{L} \prod_{j=1}^{M} (x_j - \mu_k) \right\} \triangle(x_1, \ldots, x_M) \triangle(\mu_1, \ldots, \mu_k). \tag{34}$$

Now we insert equations (33) and (34) in the formula for the correlation function $\mathcal{K}_N(\hat{\epsilon}, \hat{\mu})$ (equation (29)). It gives

$$\mathcal{K}_N(\hat{\epsilon}, \hat{\mu}) = \frac{N!}{(N-M)!M!} \left(\prod_{j=1}^{M} \epsilon_j^{M-N} \right) \frac{Z_{N-M}}{Z_N} \frac{1}{\triangle(\hat{\lambda}, \hat{\mu})} J_M(\hat{\epsilon}, \hat{\mu}) \tag{35}$$

where $J_M(\hat{\epsilon}, \hat{\mu})$ is the M-fold integral,

$$J_M(\hat{\epsilon}, \hat{\mu}) = \int \mathrm{d}x_1 \ldots \mathrm{d}x_m \det[\Phi_i(x_j)]_{1\leqslant i,j\leqslant M} \det[\pi_{N-M+j-1}(r_i)]_{1\leqslant i,j\leqslant L+M} \tag{36}$$

where we have introduced M functions $\Phi_i(x)$,

$$\Phi_i(x) = \frac{x^{i-1} G(x)}{\prod_{j=1}^{M} (\epsilon_j - x)} \qquad G(x) = \mathrm{e}^{-NV(x)} x^{N-M}. \tag{37}$$

Thus the computation of the correlation function $\mathcal{K}_N(\hat{\epsilon}, \hat{\mu})$ is reduced to that of the integral of M variables. This integral can be further transformed to a determinant form. To proceed we first note that $\det[\Phi_i(x_j)]$ can be simplified[2] as

$$\det[\Phi_i(x_j)] = \frac{1}{\triangle(\hat{\epsilon})} \det\left(\frac{G(x_i)}{\epsilon_j - x_i}\right). \tag{41}$$

Inserting expression (41) in integral (36) and rewriting the determinants as sums over permutations, we find that

$$J_M(\hat{\epsilon}, \hat{\mu}) = \frac{M!}{\triangle(\hat{\epsilon})\triangle(\hat{\mu})} \det \begin{vmatrix} \tilde{\pi}_{N-M}(\epsilon_1) & \dots & \tilde{\pi}_{N+L-1}(\epsilon_1) \\ \vdots & & \\ \tilde{\pi}_{N-M}(\epsilon_M) & \dots & \tilde{\pi}_{N+L-1}(\epsilon_M) \\ \pi_{N-M}(\mu_1) & \dots & \pi_{N+L-1}(\mu_1) \\ \vdots & & \\ \pi_{N-M}(\mu_L) & \dots & \pi_{N+L-1}(\mu_L) \end{vmatrix} \tag{42}$$

where

$$\tilde{\pi}_k(\epsilon) = \int \frac{\mathrm{e}^{-NV(x)} x^{N-M}}{\epsilon - x} \pi_k(x)\,\mathrm{d}x \qquad k \in [N-M, N-M+L-1]. \tag{43}$$

Using the orthogonality of monic polynomials $\pi_k(x)$ with respect to the measure $\mathrm{e}^{-NV(x)}\,\mathrm{d}x$, we observe the relation between $\tilde{\pi}_k(\epsilon)$ and the Cauchy transforms $h_k(\epsilon)$:

$$\tilde{\pi}_k(\epsilon) = -2\pi \mathrm{i} \epsilon^{N-M} h_k(\epsilon). \tag{44}$$

We insert expressions (42) and (44) in formula (35) for the correlation function $\mathcal{K}_N(\hat{\epsilon}, \hat{\mu})$. Using the relations between coefficients Z_j, c_j, γ_j, we finally prove (8).

5. Correlation function $\mathcal{K}_N(\hat{\epsilon}, \hat{\mu})$ and the Riemann–Hilbert problem for orthogonal polynomials

Our formula (8) enables us to express the correlation function $\mathcal{K}_N(\hat{\epsilon}, \hat{\mu})$ in terms of solutions of Riemann–Hilbert problems for orthogonal polynomials proposed by Fokas, Its and Kitaev [15, 16]. It follows from (8) that the correlation function $\mathcal{K}_N(\hat{\epsilon}, \hat{\mu})$ is determined by monic orthogonal polynomials and their Cauchy transforms. In turn, the monic orthogonal polynomials and their Cauchy transforms can be understood as elements of a (matrix-valued) solution of the following Riemann–Hilbert problem. Let contour Σ be the real line oriented

[2] This follows from the algebraic relation

$$\frac{x_j^{i-1} G(x_j)}{\prod_{l=1}^M (\epsilon_l - x_j)} = (-)^M \sum_{K=1}^M \frac{\epsilon_k^{i-1}}{\prod_{l\neq k}(\epsilon_k - \epsilon_l)} \frac{G(x_j)}{x_j - \epsilon_k} \tag{38}$$

which can be derived from identity (11). With the above equation we have

$$\det[\Phi_i(x_j)] = (-)^{M^2} \det\left(\frac{G(x_i)}{\epsilon_j - x_i}\right) \det\left(\epsilon_j^{i-1} \prod_{l\neq j} \frac{1}{\epsilon_j - \epsilon_l}\right). \tag{39}$$

Noting that

$$(-)^{M^2} \det\left(\epsilon_j^{i-1} \prod_{l\neq j} \frac{1}{\epsilon_j - \epsilon_l}\right) = \frac{1}{\triangle(\hat{\epsilon})} \tag{40}$$

we obtain equation (41).

from left to right. The upper side of the complex plane with respect to the contour will be called the positive one and the down side the negative one. Once an integer $n \geqslant 0$ is fixed the Riemann–Hilbert problem is to find a 2×2 matrix-valued function $Y = Y^{(n)}(z)$ satisfying the following conditions:

- $Y^{(n)}(z)$ – analytic in $\mathrm{C}\backslash\Sigma$
- $Y_+^{(n)}(z) = Y_-^{(n)}(z)\begin{pmatrix}1 & \mathrm{e}^{-nV(z)}\\ 0 & 1\end{pmatrix}$, $z \in \Sigma$
- $Y^{(n)}(z) \mapsto (I + \mathcal{O}(z^{-1}))\begin{pmatrix}z^n & 0\\ 0 & z^{-n}\end{pmatrix}$ as $z \mapsto \infty$.

Here $Y_\pm^{(n)}(z)$ denotes the limit of $Y^{(n)}(z')$ as $z' \mapsto z \in \Sigma$ from the positive/negative side. As is proved by Fokas, Its and Kitaev [15, 16] the solution of the Riemann–Hilbert problem is unique and is given by

$$Y^{(n)}(z) = \begin{pmatrix}\pi_n(z) & h_n(z)\\ \gamma_{n-1}\pi_{n-1}(z) & \gamma_{n-1}h_{n-1}(z)\end{pmatrix} \qquad \mathrm{Im}\, z \neq 0. \tag{45}$$

On comparing formulae (8) and (45) we observe that the correlation function $\mathcal{K}_N(\hat{\epsilon}, \hat{\mu})$ can be expressed in terms of the matrix elements of the solution of the above Riemann–Hilbert problem. The relation provides us with a possibility to investigate the asymptotics of $\mathcal{K}_N(\hat{\epsilon}, \hat{\mu})$ at large N for essentially any potential function $V(x)$ entering the probability distribution (3). The details will be presented in a forthcoming publication [24]. Here we just outline the main steps. First we will show that formula (8) can be rewritten as a determinant whose entries are two kernel functions. Those kernel functions are expressible in terms of the solution of the Riemann–Hilbert problem for orthogonal polynomials. The large N asymptotics in the Dyson scaling limit can be studied by the steepest descent/stationary phase method for Riemann–Hilbert problems introduced by Deift and Zhou [19] and developed further in [20, 21]. As a result we will prove the universality of various quantities related to $\mathcal{K}_N(\hat{\epsilon}, \hat{\mu})$. In particular, we shall be able to prove the universality of the distributions of the level curvatures and the local density of states, as well as derive the Poisson kernel distribution of S-matrix in a random matrix model of quantum chaotic scattering. Understanding of universality of objects arising in random matrix theory in various scaling limits was recently a subject of intensive work both in physical [26] as well as mathematical [27, 17, 22, 23] communities.

6. 'Duality relations' for the Gaussian case

For the particular case of the Gaussian measure $V(x) = x^2/2$ equation (5) for the ensemble average of products of characteristic polynomials can be written in an equivalent form:

$$\int_{N\times N} \mathrm{d}[\hat{H}]\,\mathrm{e}^{-\frac{N}{2}\mathrm{Tr}\,\hat{H}^2}\det(\hat{\lambda}\otimes\mathbf{1}_N - \mathbf{1}_L\otimes\hat{H}) \propto \int_{L\times L}[\mathrm{d}\hat{Q}]\,\mathrm{e}^{-\frac{N}{2}\mathrm{Tr}\,\hat{Q}^2}\det(\hat{\lambda} - \mathrm{i}\hat{Q})^N. \tag{46}$$

Here the integration on the left-hand side goes over the manifold of Hermitian matrices of the size $N \times N$, whereas on the right-hand side it goes over $L \times L$ Hermitian matrices.

This integral identity can be viewed as a certain 'duality' relation for matrix integrals. It emerged in various physical contexts, most notably in the context of matrix models of the string theory (see, e.g., [28], p 27) where it played an important role in understanding equivalence between one-matrix models of the quantum gravity and the so-called Kontsevich-type models [28, 29]. To understand equation (46) one recalls that the orthogonal polynomials for the Gaussian case are Hermite polynomials possessing an integral representation:

$$\pi_k(\lambda) \propto \mathrm{e}^{N\lambda^2/2}\int_{-\infty}^{\infty}\mathrm{d}q\, q^k\,\mathrm{e}^{-N(\frac{q^2}{2}-\mathrm{i}\lambda q)}. \tag{47}$$

Substituting (47) into (5) one easily brings the right-hand side of the latter to the form

$$\left\langle \prod_{j=1}^{L} \mathcal{Z}_N[\lambda_j, H] \right\rangle_H \propto \frac{\mathrm{e}^{\frac{N}{2}\mathrm{Tr}\,\hat{\lambda}^2}}{\Delta\{\lambda\}} \int \mathrm{d}\hat{Q}_\lambda \Delta\{\hat{Q}_\lambda\}\, \mathrm{e}^{-N[\frac{1}{2}\mathrm{Tr}\,\hat{Q}_\lambda^2 - \mathrm{i}\,\mathrm{Tr}\,\hat{Q}_\lambda\hat{\lambda}]} \det \hat{Q}_\lambda^N \tag{48}$$

where $\hat{Q}_\lambda = \mathrm{diag}(q_1, \ldots, q_L)$. On the other hand, the above expression can be obtained after shifting $\lambda - \mathrm{i}\hat{Q} \to -\mathrm{i}\hat{Q}$ in equation (46), diagonalizing the $L \times L$ matrix $\hat{Q}$ and integrating the angular degrees of freedom with the help of the Itzykson–Zuber–Harish–Chandra formula. A particular case of formula (48) was mentioned in [5] in the context of symmetric Jack polynomials.

It is easy to check that the Cauchy transforms of the Hermite polynomials are given by

$$h_k(\epsilon) \propto \int \mathrm{d}q\, q^k\, \mathrm{e}^{-N(\frac{q^2}{2} - \mathrm{i}\,\mathrm{sgn}\,(\mathrm{Im}\,\epsilon)\epsilon q)}. \tag{49}$$

Here the integration domain is $0 < q < \infty$ for $\mathrm{Im}\,\epsilon > 0$ and $-\infty < q < 0$ for $\mathrm{Im}\,\epsilon < 0$. This gives us a possibility to rewrite our main object—correlation function (8)—in the form of $(M + L)$-fold integral:[3]

$$\mathcal{K}_N(\hat{\epsilon}, \hat{\mu}) \propto \frac{\mathrm{e}^{\frac{N}{2}\mathrm{Tr}\,\hat{\mu}^2}}{\Delta\{\mu\}\Delta\{\epsilon\}} \int \mathrm{d}\hat{Q}\Delta\{\hat{Q}\}\, \mathrm{e}^{-N[\frac{1}{2}\mathrm{Tr}\,\hat{Q}^2 - \mathrm{i}\,\mathrm{Tr}\,\hat{Q}\hat{E}]} \det \hat{Q}^{N-M} \tag{50}$$

where $\hat{Q} = (q_1, \ldots, q_L, q_{L+1}, \ldots, q_{M+L})$ and $\hat{E} = (\hat{\mu}, \hat{\epsilon}_+, -\hat{\epsilon}_-)$, with ϵ_+ and ϵ_- denoting spectral parameters ϵ_k with positive (negative) imaginary part, respectively. The integration domain is $-\infty < q_l < \infty$ for $1 \leqslant l \leqslant L$ but $0 < q_l < \infty$ or $-\infty < q_l < 0$ for $L < l \leqslant M + L$, depending on the sign of the imaginary part of the corresponding spectral parameter.

Expression (50) is a generalization of duality relation (48). It arises most naturally in the method based on Gaussian integral representations and Itzykson–Zuber type integrations [11]. Similar identities hold for the chiral ensemble (Laguerre polynomials) (see [13] for more details).

7. Conclusions

In this paper we have found an exact formula for the general correlation function $\mathcal{K}_N(\hat{\epsilon}, \hat{\mu})$ containing both products and ratios of characteristic polynomials. Our result is valid for an arbitrary ensemble of Hermitian matrices of $\beta = 2$ class. The formula obtained establishes a correspondence between the correlation function $\mathcal{K}_N(\hat{\epsilon}, \hat{\mu})$ and the Riemann–Hilbert problem for orthogonal polynomials. It is remarkable as it enables us to study the large N asymptotics of this correlation function via the Riemann–Hilbert approach. Among interesting prospects for future research we would like to mention a challenging problem of extending our calculations to other symmetry classes $\beta = 1, 4$ (cf [31]) as well as to ensembles of non-Hermitian random matrices important for the problems of quantum chaotic scattering [32].

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[3] The formula equivalent to the simplest case of our equation (50) with $L = M = 1$ appeared in [30].

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